Renormalization group analysis of the three-dimensional Gross–Neveu model at finite temperature and density

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Abstract

The Renormalization Group flow equations obtained by means of a proper time regulator are used to analyze the restoration of the discrete chiral symmetry at non-zero density and temperature in the Gross–Neveu model in $d = 2 + 1$ dimensions. The effects of the wave function renormalization of the auxiliary scalar field on the transition have been studied. The analysis is performed for a number of fermion flavors $N_f = 12$ and the limit of large $N_f$ is also considered. The results are compared with those coming from lattice simulations.

In recent years there has been a renewed interest in the chiral symmetry restoration in QCD in connection with its possible experimental observation in the transition from the confined to the quark–gluon plasma phase. Lattice simulations, which are the most relevant tool to study such nonperturbative phenomena, when employed to investigate QCD at finite baryon density, must deal with the main problem of the appearance of complex terms in the Euclidean action due to the presence of the chemical potential and only recently new lattice techniques have been proposed [1,2], which overcome this problem, at least for not too large values of the chemical potential. On the other hand, the chiral symmetry breaking at finite temperature and density has been investigated in other models [3–7], which have a well defined action even for non-vanishing chemical potential and which, despite their simplicity, give a qualitative picture of the transition that could provide reasonable indications of this phenomenon in the full QCD framework.

The simplest model is represented by the Gross–Neveu model [8], characterized by either discrete or continuous chiral symmetry, which in three spacetime dimensions shows a critical coupling $g^*$ that indicates the threshold for the occurrence of chiral symmetry breaking at zero temperature and density. This model has an interacting continuous limit with a spectrum that contains composite fermion and antifermion states analogous to baryons and mesons and its four fermion interaction can be regarded as an effective interaction...
for quarks at intermediate energies and at not very high temperatures where the effect of heavy mesons is still suppressed. Moreover, in three dimensions the model is renormalizable order by order in the $1/N$ expansion around the critical coupling $g^*$ [9].

In addition to lattice calculations, another non-perturbative tool, the Exact Renormalization Group (ERG) (see [10] for a review of the subject and a more complete list of references) has been employed to study four fermion models, such as the Nambu–Jona-Lasinio model at finite temperature and density analyzed in [11], as well as the fixed point structure of the Gross–Neveu model in [12]. In this Letter we make use of the Renormalization Group to investigate the specific problem of the phase diagram of the tridimensional Gross–Neveu model in [12]. In this Letter we make use of the Renormalization Group to investigate the specific problem of the phase diagram of the tridimensional Gross–Neveu model in [12]. In this Letter we make use of the Renormalization Group to investigate the specific problem of the phase diagram of the tridimensional Gross–Neveu model in [12]. In this Letter we make use of the Renormalization Group to investigate the specific problem of the phase diagram of the tridimensional Gross–Neveu model in [12]. In this Letter we make use of the Renormalization Group to investigate the specific problem of the phase diagram of the tridimensional Gross–Neveu model in [12]. In this Letter we make use of the Renormalization Group to investigate the specific problem of the phase diagram of the tridimensional Gross–Neveu model in [12]. In this Letter we make use of the Renormalization Group to investigate the specific problem of the phase diagram of the tridimensional Gross–Neveu model in [12]. In this Letter we make use of the Renormalization Group to investigate the specific problem of the phase diagram of the tridimensional Gross–Neveu model in [12].

This latter version of the flow equation, often indicated as Proper Time Renormalization Group (PTRG) is typically obtained as an improvement of the one loop effective action by making use of the proper time representation [16] and introducing a suitable multiplicative cutoff function as a regulator of the infrared modes [17,18]. The PTRG does not belong to the class of ERG flows which can be formally derived from the Green’s functions generator without resorting to any truncation or approximation [19–21]. Nevertheless the PTRG flow equation has the property of preserving the symmetries of the theory and, in addition, it has a quite simple and manageable structure. Moreover, the PTRG flow, although not able to fully reproduce perturbative expansions beyond the one loop order [22], provides excellent results when used to evaluate the critical properties such as the critical exponents of the three-dimensional scalar theories at the nongaussian fixed point [23,24]. In particular the determination is optimized when one takes the sharp limit of the cutoff function on the proper time and it turns out to be much more accurate than the one corresponding to a smoother regulator. The PTRG also provides excellent determinations of the energy levels of the quantomechanical double well [25]. These results are obtained at the first order in the derivative expansion of the effective action, i.e., by reducing the full flow equation to two coupled differential equations for the potential and the wave function renormalization, and in our opinion this is a clear indication that, within this specific approximation, the PTRG flow is reliably accurate. According to this point of view it is interesting to test the PTRG flow in a different context which involves fermionic systems at finite temperature and density and in particular to compare the results coming from this approach with those obtained from the lattice investigations.

Therefore in the following we focus on the Gross–Neveu model with Euclidean action

$$S^{(0)} = \int d^3x \left( i \bar{\psi}_j \gamma \psi_j - \frac{g^2}{2N_f} (\bar{\psi}_j \psi_j)^2 \right),$$

(1)

where $\gamma^\mu \partial_\mu$ involves three gamma matrices ($\mu$ takes the three values: 1, 2, 4), $\psi_j$ is a four component spinor and $j$ is the flavor index: $j = 1, \ldots, N_f$. The action (1) is invariant under the discrete chiral transformations $\psi_j \rightarrow e^{i\theta} \psi_j$, $\gamma_j \rightarrow -\bar{\psi}_j \gamma^5$. As it is well known [8], instead of considering Eq. (1), it is helpful to introduce an auxiliary scalar field $\sigma$ and put the action in a bosonized form. In fact after adding to $S^{(0)}$ the term $\int d^3x (1/2)(\sigma + \bar{\psi}_j \psi_j)/\sqrt{N_f}$, which is quadratic in $\sigma$ and therefore does not modify the fermionic theory, one alternatively describes the model by means of the action

$$S = \int d^3x \left( \frac{1}{2} \sigma^2 + i \bar{\psi}_j \gamma^5 \psi_j + \frac{\bar{\psi}_j \psi_j}{\sqrt{N_f}} \right).$$

(2)

The action (2) is to be taken as the bare action which fixes the ultraviolet boundary conditions of the renormalization group flow. In fact, in the flow equations we consider a more general structure of the effective action which includes a scalar potential $U_k(\sigma)$, the kinetic term for the scalar sector, properly normalized by the wave function renormalization $Z_k$ and also a fermionic wave function renormalization $Z_k^\psi$:

$$S_k = \int d^3x \left( \frac{Z_k}{2} \partial_\mu \sigma \partial^\mu \sigma + U_k(\sigma^2) 
+ i Z_k^\psi \bar{\psi}_j \gamma^5 \psi_j + \frac{\bar{Z}_k}{\sqrt{N_f}} \sigma \bar{\psi}_j \psi_j \right).$$

(3)
In Eq. (3) we have explicitly introduced a dependence of the various terms on the running momentum scale $k$ which parametrizes the PTRG flow. In fact the renormalization group equations describe their evolution as functions of $k$, starting from the ultraviolet boundary conditions given in Eq. (2), down to the far infrared region, when all modes have been integrated out. So the ultraviolet boundary conditions, corresponding the large value $k = \Lambda$, are $Z_A = 0$, $Z_A^\nu = 1$, $U_A(\sigma^2) = \sigma^2/2$, and a fixed value for $\hat{g}_A$. We shall limit ourselves to field independent wave function renormalizations $Z_k$ and $Z_k^\nu$ and to a potential which is polynomial in $\sigma^2$.

The shape of effective potential $U_k(\sigma^2)$ in the infrared region, i.e., at $k \sim 0$, does provide informations about the chiral symmetry breaking and restoration, which are in fact related to the expectation value of the auxiliary scalar field $\sigma$. If this expectation value is nonvanishing then the fermion becomes massive, whereas below $\hat{g}^*$ the field $\sigma$ has zero expectation value and the fermion is massless, preserving chirality. When $\hat{g} > \hat{g}^*$ a sufficient increase of the temperature and (or) of the density is expected to restore the symmetry.

A saddle point expansion of Eq. (2) yields the one loop effective action

$$S_{UL} = S - \frac{1}{2} \text{Tr} \ln(A^*A) + \frac{1}{2} \text{Tr} \ln \left[ \frac{\delta^2 S}{\delta \sigma \delta \sigma} - 2 \frac{\delta^2 S}{\delta \sigma \delta \psi} A^{-1} \frac{\delta^2 S}{\delta \psi \delta \sigma} \right],$$

where

$$A = \frac{\delta^2 S}{\delta \psi \delta \psi}.$$  \hspace{1cm} (5)

The logarithms in Eq. (4) can be expressed by means of the proper time representation

$$\ln(X) = - \int_0^\infty \frac{ds}{s} e^{-sX}$$

and the infrared modes are properly regularized by means of a cutoff function $f_k$, [18], multiplicatively introduced into Eq. (6) by replacing $e^{-sX}$ with $e^{-sX} f_k$.

Clearly this replacement introduces a dependence on the variable $k$ in Eq. (4). Then the PTRG equations are obtained by replacing everywhere in Eq. (4) the action with the $k$ dependent ansatz $S_k$ of Eq. (3) and by taking the derivatives with respect to $k$ of each side of Eq. (4). Typically $f_k$ is taken as a smooth heat kernel cutoff [13–15,18], however as shown in [23,24] the PTRG equations are optimized when the sharp limit of $f_k$ is taken, or, more precisely when its derivative with respect to $k$ (which is the relevant quantity in the differential flow equations) is taken as a delta-function [20,22]

$$k \partial_k f_k = -2s\delta \left( s - \frac{1}{k^2} \right).$$

Therefore in the following we take $f_k$ as implicitly defined in Eq. (7) and the PTRG flow equation has a particularly simple form

$$k \partial_k S_k = \text{Tr} \exp \left[ \frac{S''_{k,(\sigma \sigma)} - 2S''_{k,(\sigma \psi)} A^{-1} S''_{k,(\psi \sigma)}}{Z_k^\nu k^2} \right]$$

$$- \text{Tr} \exp \left[ \frac{-A^*_k A_k}{(Z_k^\nu)^2 k^2} \right],$$

where we used a concise notation for the second derivatives of the action and $Z_k$ and $Z_k^\nu$ provide the correct normalization of the two exponential terms. The traces are extended to all the relevant degrees of freedom.

The extension of the flow equation to finite temperature is done following the standard procedure [26], i.e., by imposing periodic boundary conditions on the Euclidean time component with period fixed by the inverse temperature $1/T$ and consequently discretizing the corresponding component of the momentum. Therefore the integration on the component $q_4$ (the index 4 is taken in analogy to the one used for the gamma matrix $\gamma^4$) is to be converted into the infinite sum over the Matsubara frequencies, which are respectively $q_4 \rightarrow \omega_n = 2\pi n T$ for bosons and $q_4 \rightarrow \nu_n = (2n + 1)\pi T$ for fermions and $n \in \mathbb{Z}$, $\int dq_4 \rightarrow 2\pi T \sum_n$.

Finally, finite fermion density effects are accounted for by adding to the bare action the chemical potential $\mu$ coupled to the density $\bar{\psi} \gamma^4 \psi$. Then we can compute the purely fermionic contribution to Eq. (8) in the
momentum space
\[ A_k^A = (Z_k^\psi)^2 (q^2 + v_n^2) + \mu^2 + g_k^2 \sigma^2 - 2i \mu (Z_k^\psi q + g_k \sigma) v^4, \]
where \( N_f \) has been absorbed into the coupling constant \( g_k \sqrt{N_f} = \tilde{g}_k \).

We are now able to extract from the full flow equation (8) the set of coupled equations for the various \( k \) dependent terms of the effective action. The equation for the potential \( U_k \) is simply obtained by considering constant field configurations for \( \sigma \) and vanishing fermion fields in Eq. (8) and, by performing the integral on the momentum, we get
\[ k \partial_k U_k = \Omega_b + \Omega_f, \]
where the boson contribution, after performing the momentum integrals, is
\[ \Omega_b = \frac{T k^2}{4\pi} \sum_n \exp \left[ -\frac{\omega_n^2}{k^2} - M_b Z_k k^2 \right] \]
with \( M_b = 2U_k + 4\sigma^2 \tilde{U}_k \) (the dots indicate derivatives with respect to \( \sigma^2 \)), and the fermion contribution is
\[ \Omega_f = -2N_f T \sum_n \int \frac{d^2 q}{(2\pi)^2} \exp \left[ -\frac{v^2_n}{k^2} \right] \times \left\{ \exp \left[ -\frac{(\mu - \sqrt{Z_k^\psi q^2 + g_k^2 \sigma^2})^2}{(Z_k^\psi)^2} \right] + \exp \left[ -\frac{\left( \mu + \sqrt{Z_k^\psi q^2 + g_k^2 \sigma^2} \right)^2}{(Z_k^\psi)^2} \right] \right\}. \]

Equation (12) shows the fermion contribution in a compact form. However, the momentum integrals can be performed analytically which makes easier the numerical integration of the set of flow equations.

In order to derive the equation for \( Z_k^\psi \) one has to expand the exponentials in the right-hand side of Eq. (8) retaining the commutators of the coordinate and momentum dependent terms, and collecting the full coefficient of \( \partial_q \sigma \partial_q \sigma \). This procedure is already discussed in [27] and, specifically for the PTRG flow equation of \( Z_k \) in scalar theories, in [22–24]. Moreover, here we are only interested in the leading corrections to the flow of the potential \( U_k \) and therefore, instead of solving the problem by taking \( Z_k \) as well as \( Z_k^\psi \) and \( g_k \) as arbitrary functions of the fields, we limit ourselves to consider the particular value of these parameters when computed at the minimum of the potential at each value of \( k \), \( \sigma = \tilde{\sigma}_k \) and for vanishing fermion fields. The lowest order equation for \( Z_k \) is therefore:
\[ k \partial_k Z_k = \frac{T k^2}{2\pi} \sum_n \exp \left[ -\frac{\omega_n^2}{k^2} - M_b \right] \frac{1}{Z_k k^2} \left( \frac{2g_k^2 Z_k (M_b)^2}{3(Z_k k^2)^2} \right) \]
\[ \times \left( \frac{1 - 4g_k^2 \tilde{\sigma}_k^2}{3(Z_k k^2)^2} \right). \]

The evolution equation of \( Z_k^\psi \) is obtained by taking the coefficient of \( \bar{\psi}_j \gamma^\lambda \psi_j \) after expanding the exponentials in the right-hand side of Eq. (8):
\[ k \partial_k Z_k^\psi = \frac{2g_k^2}{Z_k k^2} T \sum_n \int \frac{d^2 q}{(2\pi)^2} \exp \left[ -\frac{v_n^2}{k^2} \right] \times \left\{ \frac{Z_k (q^2 + \omega_n^2) + M_b}{Z_k k^2} \times \left( \frac{2Z_k^\psi (q^2 + v_n^2) + M_b}{Z_k k^2} \frac{W^2 - 4Z_k^\psi^2 \mu^2 v_n^2}{2W^2 + 4Z_k^\psi^2 \mu^2 v_n^2} \right) \right\} \times \left( \frac{W}{W^2 + 4Z_k^\psi^2 \mu^2 v_n^2} \right), \]
and the one for \( g_k \) by selecting the coefficient of \( \sigma \partial_q \sigma \partial_q \sigma \):
\[ k \partial_k g_k = \frac{2g_k^3}{Z_k k^2} T \sum_n \int \frac{d^2 q}{(2\pi)^2} \exp \left[ -\frac{v_n^2}{k^2} \right] \times \frac{Z_k (q^2 + \omega_n^2) + M_b}{Z_k k^2} \times \left( \frac{W}{W^2 + 4Z_k^\psi^2 \mu^2 v_n^2} \right), \]
where \( M_b \) and \( M_b \) in Eqs. (13)–(15) are evaluated at \( \sigma = \tilde{\sigma}_k \) and \( W \) is defined as
\[ W = Z_k^\psi (q^2 + v_n^2) + g_k^2 \tilde{\sigma}_k^2 - \mu^2. \]

However, in the following we shall not solve the full potential equation (10) which is a partial differential equation, but rather we treat a set of ordinary differential equations for the various coefficients of...
a truncated polynomial expansion of the potential. In fact, according to [14] such truncations converge quite rapidly and we choose to retain powers of the field up to $\sigma^4$ in the expansion. Therefore, we parametrize the potential, expanded around the local minimum $\bar{\sigma}_k$, in the following way

$$U_k(\sigma^2) = u_{0,k} + \frac{1}{2} u_{2,k} (\sigma^2 - \bar{\sigma}_k^2) + \sum_{i=2}^{4} u_{2i,k} (\sigma^2 - \bar{\sigma}_k^2)^i. \quad (17)$$

The parametrization in Eq. (17) contains five $k$ dependent parameters. In fact when the minimum of the potential is located at $\bar{\sigma}_k = 0$, then the independent parameters are $u_{0,k}, u_{2,k}, u_{4,k}, u_{6,k}, u_{8,k}$. When instead the minimum is at a nonvanishing $\bar{\sigma}_k$, one has $\dot{U}_k(\bar{\sigma}_k^2) = 0$ and then $u_{2,k} = 0$ identically and the $k$ dependent parameters are still five with $u_{2,k}$ replaced by $\bar{\sigma}_k$. For our potential at $k = \Lambda$ we have $u_{2,\Lambda} = 1$ and the minimum at $\bar{\sigma}_\Lambda = 0$ and therefore, we start our flow by explicitly considering the evolution of $u_{2,k}$. Then, when $k$ is lowered, $u_{2,k}$ diminishes and (only for suitable values of $T$ and $\mu$) eventually vanishes at $\bar{k}$ (with $0 < \bar{k} < \Lambda$). So, at $k = \bar{k}$, we replace the flow equation for $u_{2,k}$ with the one for $\bar{\sigma}_k$, with the initial condition $\bar{\sigma}_k = 0$ and require continuity for the other parameters $u_{0,k}, u_{4,k}, u_{6,k}, u_{8,k}$.

The flow equations for the parameters in Eq. (17) are directly obtained from Eq. (10) by repeated differentiations with respect to $\sigma^2$ (we shall not display them explicitly) and the one for $\bar{\sigma}_k$ is instead derived from the minimum condition,

$$(2 \bar{\sigma}_k \dot{U}_k (\bar{\sigma}_k^2)) k \partial_k \bar{\sigma}_k = - k \partial_k \dot{U}_k (\bar{\sigma}_k^2). \quad (18)$$

Incidentally we note that $u_{0,k}$ never appears in the other equations and therefore this parameter does not affect the evolution of the others.

As a first step in the analysis of the flow equations, we briefly review the well-known case $N_f \to \infty$, that can be treated analytically. In fact, since the minimum of the potential increases as $\sqrt{N_f}$ while the generated mass of the fermion does not depend on $N_f$ (we recall that in our definitions the coupling $g_k$ is proportional to $1/\sqrt{N_f}$) it is convenient to rescale the fields as $\sigma \to \sqrt{N_f} \sigma$, $\psi \to \sqrt{N_f} \psi$ as well as the potential $U_k \to N_f U_k$ so that the action, expressed in terms of these new variables and of $\dot{g}_k$, is homogeneously rescaled by the factor $N_f$. It is easy to realize that after these changes the right-hand sides of the flow equations for $Z_k$ and $\dot{g}_k$ are suppressed in the limit $N_f \to \infty$ and these parameters remain fixed to their initial value at $k = \Lambda$ and in particular $Z_k^\psi = 1$. Also, in Eq. (10) the term $\bar{\sigma}_k$ is suppressed and the equation for the potential depends only on the fermionic contribution which does not contain any dependence on the potential itself or on its derivatives or on the parameter $Z_k$. As a consequence, the equations for $Z_k$ and for $U_k$ decouple. We neglect the former and focus on the latter, which is relevant for the symmetry breaking problem, and we note that in this limit Eq. (10) is no longer a differential equation and can be simply integrated in $k$ from zero to infinity with the help of Eq. (6) and by recalling the boundary condition for the potential in the ultraviolet region assigned in Eq. (2). By extending the $k$ integration up to infinity, we have implicitly removed the ultraviolet scale $\Lambda$ and this will introduce some divergences in our computation which at some point need to be regulated. However, in this way we can more easily make contact with the standard calculations of the large $N_f$ case (see, e.g., [28]). In order to determine the extrema of the potential it is convenient to focus on its derivative with respect to $\sigma$ and, after performing the sum over the Matsubara frequencies [26], we get (we neglect for simplicity the subscript $k$ for the coupling $\dot{g}$ since here, as noticed above, it is $k$ independent)

$$\frac{\partial U_{k=0}}{\partial \sigma} = \sigma - \int \frac{d^2 q}{(2\pi)^2} \frac{\dot{g}_k^2 \sigma}{\sqrt{q^2 + \dot{g}_k^2 \sigma^2}} \times \left[ \tanh \frac{\sqrt{q^2 + \dot{g}_k^2 \sigma^2} + \mu}{2T} + \tanh \frac{\sqrt{q^2 + \dot{g}_k^2 \sigma^2} - \mu}{2T} \right]. \quad (19)$$

In the particular case $\mu = T = 0$, if we regulate in Eq. (19) the divergent integral on the momentum $q$ with the ultraviolet cutoff $\Lambda$, we get the gap equation that defines the fermion mass at zero temperature and chemical potential [28]

$$\frac{\pi}{g^2} = \Lambda - \dot{g}_\sigma = \Lambda - m_f. \quad (20)$$

If we consider $T = 0$ and finite chemical potential we see that the hyperbolic tangents in Eq. (19) in the
limit of vanishing temperature become step functions and, by making use of the gap equation (20) to get rid of the regulator $A$, it is easy to check that taking $\mu < m_f$ or $\mu > m_f$ yields two different behaviors. In fact, in the former case the same gap equation as for $\mu = 0$ is obtained, in the latter case we get the extremum condition $\hat{g}^2\sigma(\mu - m_f) = 0$ that has the unique solution $\sigma = 0$. Therefore, at $T = 0$ a first order transition is observed when the chemical potential is equal to $m_f$.

Finally when also $T \neq 0$, again with the help of Eq. (20), we get the gap equation

$$\left(\cosh \frac{\hat{g}\sigma}{T} - \frac{e^{(m_f/T)}}{2} + \cosh \frac{\mu}{T}\right)\sigma = 0. \quad (21)$$

For nonvanishing $T$, Eq. (21), in addition to $\sigma = 0$, has another solution only for sufficiently small chemical potential $\mu < \bar{\mu}(T)$ (with $\bar{\mu}(T) \sim m_f$ for small $T$). This solution tends to zero for $\mu \to \bar{\mu}(T)$ from below and disappears for $\mu > \bar{\mu}(T)$ where the only extremum is $\sigma = 0$. Then the transition in this case is second order. Only at $T = 0$ there is a singular behavior that leads to a first order transition [5].

Let us now consider the finite $N_f$ case and, in order to test the PTERG we shall compare our analysis with the lattice results obtained in [5] where $N_f = 12$ is taken to observe the leading corrections to the $N_f \to \infty$ case. Therefore, in the following we select this particular value of $N_f$ and solve numerically our flow equations. The flow starts at the scale $k = \Lambda$ and ends at $k = 0$ and we varied $\Lambda$ in the range $10^3 - 10^5$ to test the insensitivity of the final output of the dimensionless parameters to the specific value of the ultraviolet scale. The initial values of the parameters at $k = \Lambda$ are those indicated below Eq. (3) and it must be noted that $u_{2,\Lambda} = 1$, independently of the particular value of $\Lambda$ and this implies that $u_{2,k}$ must be dimensionless. Consequently all the canonical dimensions of the other parameters must be arranged according to this point.

The most simple version of the flow equations is obtained by a further reduction of the number of running parameters and, specifically, by neglecting the flow of the renormalization $Z_k$ of the scalar field. Since in our problem the bare value of this parameter is $Z_{\Lambda} = 0$, this approximation corresponds to have $Z_k = 0$ at any scale $k$ and therefore, to totally neglect the kinetic contribution of the scalar field. Moreover, as it can be seen from Eqs. (14) and (15), in this case also $Z^g_k$ and $g_k$ become $k$-independent and stay fixed to their initial value and in Eq. (10), $\Omega_0 = 0$. When $Z_k$ is turned on and the full set of flow equations is considered, still $Z^g_k$ and $g_k$ do not show any significant change from their initial values. On the contrary $Z_k$ has a not negligible flow and when the scale $k$ goes from the ultraviolet to the infrared region, this parameter grows from zero to a finite value, and this was already observed in models of fermions coupled to scalars [11,29]. In general the effects of the scalar renormalization are significant, as it happens, for instance, in the quantitative determination of the critical exponents of the scalar theory or of some energy levels in quantum mechanics [24,25] and, also in this problem they will turn out to be relevant.

At $\mu = T = 0$ and with $N_f = 12$ (which is the value of $N_f$ adopted in [5]) we find a critical value of the coupling that separates the symmetric and the broken phase, which, in cutoff units, is $1/(\hat{g}^*_A\sqrt{A})^2 = 0.178(2)$. In [5] the inverse squared critical coupling, expressed in different units, is 0.98 and, by addressing the difference between these two determinations to the two different scales employed to normalize the dimensionful coupling, it follows that the ratio of these two numbers directly yields the ratio of the two scales. Then, since the value of the inverse square coupling taken in [5] to study the restoration of the chiral symmetry at finite $T$ and $\mu$ is 0.75, we can easily convert it in units of our cutoff $A$ to get the input value of the coupling in our flow equations. In our scale it is $1/(\hat{g}^*_A\sqrt{A})^2 = 0.136$. At this value of the coupling and again at $\mu = T = 0$ we have determined the fermion mass which, according to the criterion chosen in [5], will be the parameter used to normalize the dimensionful quantities in our analysis at finite $\mu$ and $T$. In particular we found $m_f/A = 0.146$ and, when $Z_k = 0$, $m_f/A = 0.145$.

The phase diagram found by means of the flow equations is shown in Fig. 1. The solid line shows the transition at $N_f \to \infty$ as obtained from Eq. (21). Except for the points at $\mu = 0$ where the temperature is raised in order to observe the symmetry restoration, all the other points are obtained by keeping a fixed temperature and changing the chemical potential $\mu$ to induce the transition and the values of the temperature considered are: $T/m_f = 0, 0.24, 0.29, 0.36, 0.48$. For the circles at $\mu = 0$ and at $T/m_f = 0.48$ the errors are...
small and they are not displayed in the figure. Notice that the circle at $\mu = 0$ is almost coincident with the lattice determination of [5] which corresponds to the star (together with its error bar) on the $T$ axis in Fig. 1. As expected the circles are in very good agreement with the mean field line and it is necessary to go beyond the local potential approximation to improve the results. The inclusion of the flow of $Z_k$ induces a significant change, reducing the values of $T$ and $\mu$ at which the transition occurs. This effect, qualitatively correct, is somehow large and the points turn out to be systematically lower than the lattice results.

Also, some uncertainties affect our determinations of the transition points in the $\mu$--$T$ plane as it is shown by the error bars associated to the points in Fig. 1. In fact, for values of $\mu$ below these bars (or, in the case of the transition at $\mu = 0$, for values of $T$ below the error bar) we find the absolute minimum of the potential at a nonvanishing vacuum expectation of the field whereas, for values above these bars the minimum corresponds to a vanishing vacuum expectation of the field and chiral symmetry is thus restored. For values of $\mu$ (and $T$) within the bars it is difficult to determine the nature of the transition, especially if the flow of $Z_k$ is included and therefore, by taking the most conservative point of view, we explicitly indicated in Fig. 1 the ranges of $\mu$ and $T$ where these uncertainties are present.

As an example, in Fig. 2 $\bar{\sigma}_{k=0}/m_f$ is displayed as a function of $T/m_f$. The solid line is obtained for $Z_k = 0$ and it shows a continuous transition and the dashed line corresponds to $Z_k \neq 0$. The latter curve has a discontinuous fall to $\bar{\sigma}_k = 0$ which should indicate a first order transition. But in the temperature range indicated by the error bar on the $x$-axis of Fig. 2 (which is the same error bar shown for the corresponding point in Fig. 1) we observe in our flow equations the appearance, at small finite $k$, of a nonvanishing minimum of the potential which however, still at finite $k$, vanishes. Then, for temperatures above the error bar in Fig. 2 the minimum of the potential corresponds to $\bar{\sigma}_k = 0$ at any value of $k$. Because of this effect that, incidentally, has already been observed in [14], we believe that it is not safe to conclude that the inclusion of the flow of $Z_k$ has changed the order of the transition which moreover would be in contrast with the expectation of a second order transition. Rather, we would argue that it is more likely that a different truncation in our flow equations, which includes more operators, like field dependent contributions to the scalar wave function renormalization, could restore the smooth behavior obtained when $Z_k = 0$ and the dashed line in Fig. 2 would continuously reach zero as it happens for the other one.
In conclusion the PTRG flow equations provide a good qualitative picture of the transition at finite temperature and chemical potential, and in particular they have the correct behavior in the limit $N_f \to \infty$. Also Fig. 1 shows that the role of $Z_k$ is certainly relevant in the determination of the critical line and a reasonable comparison with the phase diagram obtained from the lattice simulations in [5], is possible. As a further check we have computed, following [5], the exponent $\nu$ for the critical behavior around the coupling $g^*$ at $T = 0$, and we found $\nu = 1.06(4)$ when $Z_k = 0$ and $\nu = 1.19(30)$ for the full set of equations, which has to be compared to the value $\nu = 1.05(10)$ found in [5]. Again the inclusion of $Z_k$ gives a correction larger that the one expected. On the other hand, the analysis of the order of the phase transition at this level of approximation is not satisfactory, although some expected features are recovered. In our opinion the quantitative discrepancies as well as the various uncertainties discussed have to be addressed to the specific truncations and approximations made on the flow equations. We expect that the inclusion of field dependent terms in the scalar wave function renormalization (as it has been already pointed out in [15] in a different context) could significantly improve the analysis and therefore this turns out to be an essential ingredient to carry out an accurate analysis of phenomenologically realistic models.

References